The equivariant Euler characteristic of moduli spaces of curves.

E. Gorsky\*

June 4, 2009

#### Abstract

We give a formula for the  $S_n$ -equivariant Euler characteristics of the moduli spaces  $\mathcal{M}_{g,n}$  of genus g curves with n marked points.

### 1 Introduction

Consider the moduli space  $\mathcal{M}_{g,n}$  of algebraic curves of genus g with n marked points. There is a natural action of the symmetric group  $S_n$  on this space, so its homologies are representations of  $S_n$ . Let  $V_{\lambda}$  be the irreducible representations of  $S_n$ ,  $s_{\lambda}$  be the corresponding Schur polynomials and

$$H^i(\mathcal{M}_{g,n}) = \sum_{\lambda} a_{i,\lambda} V_{\lambda}$$

for some integers  $a_{i,\lambda}$ . Define the  $S_n$ -equivariant Euler characteristic of  $\mathcal{M}_{g,n}$  by the formula

$$\chi^{S_n}(\mathcal{M}_{g,n}) = \sum_{i,\lambda} (-1)^i a_{i,\lambda} s_{\lambda}.$$

Let  $p_n$  denote the *n*th elementary Newton polynomial in the infinite number of variables. Then  $\chi^{S_n}(\mathcal{M}_{g,n})$  can be also calculated by the formula

 $<sup>^*</sup>$ Partially supported by the grants RFBR-007-00593, RFBR-08-01-00110-a, NSh-709.2008.1 and the Moebius Contest fellowship for young scientists.

$$\chi^{S_n}(\mathcal{M}_{g,n}) = \sum_i (-1)^i \sum_{\sigma \in S_n} (-1)^{|\sigma|} p_1^{k_1(\sigma)} \cdot \dots \cdot p_n^{k_n(\sigma)} \cdot \operatorname{Tr}(\sigma|_{H^i(\mathcal{M}_{g,n})}),$$

where  $k_i(\sigma)$  denotes the number of cycles of length i in the permutation  $\sigma$ .

The paper is organized as follows. In Section 2 we recall and present alternative proofs for two results of [12]. Theorem 1 deals with configuration spaces of points F(X, n) on a variety X with an action of a finite group G: the formula for the  $S_n$ -equivariant Euler characteristics of F(X, n)/G is presented. Theorem 2 can be considered as a fibered version of a first one.

Theorem 1 was previously proved in [11] with the usage of results of E. Getzler ([6],[7]) on Adams operations on the motivic rings. The alternative proof presented below uses only the basic properties of the Euler characteristic and seems to be more geometric. The proof of the Theorem 2 is almost the same as in [12], here it is written in a slightly more general form to clarify the logic.

From Theorem 2 we conclude that the coefficients in the generating fuction for the  $S_n$ -equivariant Euler characteristics of the moduli spaces of pointed curves are nothing but the orbifold Euler characteristics of some configuration spaces. This phenomenon seems to be quite general, and helps a lot due to the nice behaviour of the orbifold Euler characteristic. The calculation of the desired Euler characteristics is done in the Section 3, it turns out to be very similar to the results of the [13]. The final answer is presented in the Theorem 3, where it is expressed in terms of the values of a certain combinatorial function  $N(n; l_1, \ldots, l_s)$ . The compact form for these values was also found in [13], it is presented in the Lemma 5.

The above principle can be roughly illustrated as follows. Suppose that we want to calculate the Euler characteristic of a coarse moduli space  $\mathcal{M}$  of some objects  $\mathcal{X}$ . This moduli space has a natural orbifold structure – a group  $G_p$  assigned to a point  $p \in \mathcal{M}$  is isomorphic to the group of automorphisms of a corresponding object  $\mathcal{X}_p$ . Consider the enlarged moduli space

$$\widehat{\mathcal{M}} = \{ (\mathcal{X}, g) | g \in Aut(\mathcal{X}) \}$$

with the following orbifold structure – a group assigned to a pair  $(\mathcal{X}, g)$  is equal to  $Aut(\mathcal{X})$ . Then the *orbifold* Euler characteristic of the enlarged moduli space equals to the Euler characteristic of the *coarse* moduli space:

$$\chi^{orb}(\widehat{\mathcal{M}}) = \int_{\widehat{\mathcal{M}}} \frac{1}{|Aut(\mathcal{X}_p)|} d\chi = \int_{\mathcal{M}} \frac{|Aut(\mathcal{X}_p)|}{|Aut(\mathcal{X}_p)|} d\chi = \chi(\mathcal{M}).$$

In the last section we calculate the coefficients explicitly for g=2 and compare it to the result of [11].

The author is grateful to J. Bergström, S. Gusein-Zade, M. Kazaryan and S. Lando for useful discussions.

# 2 Equivariant and orbifold Euler characteristics

Let X be a variety with an action of a finite group G. Let us denote by F(X, n) the configuration space of ordered n-tuples of distinct points on X.

### Lemma 1

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \chi(F(X, n)) = (1+t)^{\chi(X)}.$$
 (1)

**Proof.** The map  $\pi_n : F(X, n) \to F(X, n-1)$ , which forgets the last point in *n*-tuple, has fibers isomorphic to X without n-1 points. Therefore  $\chi(F(X, n)) = (\chi(X) - n + 1) \cdot \chi(F(X, n-1))$ , and

$$\chi(F(X,n)) = \chi(X) \cdot (\chi(X) - 1) \cdot \ldots \cdot (\chi(X) - n + 1).$$

The action of the group G on X can be naturally extended to the action of G on F(X, n) for all n, commuting with the natural action of  $S_n$ .

**Lemma 2** Let  $\sigma \in S_n$ . Then

$$\chi([F(X,n)/G]^{\sigma}) = \frac{1}{|G|} \sum_{g \in G} \chi(F(X,n)^{g^{-1}\sigma}).$$

**Proof.** For a point  $\mathbf{y} \in F(X, n)$  whose projection on F(X, n)/G is  $\sigma$ -invariant there exists an element  $g \in G$  such that  $\sigma \mathbf{y} = g \mathbf{y}$ . Consider the set of pairs

$$S = \{(g, \mathbf{y}) | g \in G, \mathbf{y} \in F(X, n) | \sigma \mathbf{y} = g \mathbf{y} \}$$

and its two-step projection  $S \to F(X, n) \to F(X, n)/G$ . The fiber of the first projection over a point  $\mathbf{y}$  is isomorphic to G-stabiliser of  $\mathbf{y}$  or empty, the fiber of the second projection cotaining  $\mathbf{y}$  is exactly the orbit of  $\mathbf{y}$ . Therefore the cardinality of every fiber of the composed map is equal to |G|.  $\square$ 

The following theorem was deduced in [11] from the resuts of E. Getzler ([6],[7]), here we would like to reproduce its proof in more geometric and clear form.

**Theorem 1** For any  $g \in G$  we denote by  $X_k(g)$  a subset of X consisting of points with g-orbits of length k. For example,  $X_1(g)$  is a set of g-fixed points. Then the following equation holds:

$$\sum_{n=0}^{\infty} t^n \chi^{S_n}(F(X,n)/G) = \frac{1}{|G|} \sum_{q \in G} \prod_{k=1}^{\infty} (1 + p_k t^k)^{\frac{\chi(X_k(g))}{k}}.$$
 (2)

**Proof.** Since all points in  $X_k(g)$  have g-orbit of length k, we have

$$\chi(X_k(g)/(g)) = \chi(X_k(g))/k,$$

where (g) is a cyclic subgroup in G generated by g. From the equation (1) we have

$$(1 + p_k t^k)^{\frac{\chi(X_k(g))}{k}} = \sum_{m_k=0}^{\infty} \frac{p_k^{m_k} t^{k m_k}}{(m_k)!} \chi(F(X_k(g)/(g), m_k)).$$

Therefore the coefficient at  $t^n$  in the right hand side of (2) equals to

$$\sum_{\sum km_k=n} \prod_k \left( \frac{p_k^{m_k}}{m_k!} \chi(F(X_k(g)/(g), m_k)) \right).$$

On the other hand.

$$\chi^{S_n}(F(X,n)/G) = \sum_{i} (-1)^i \sum_{\sigma \in S_n} (-1)^{|\sigma|} p_1^{k_1(\sigma)} \cdot \dots \cdot p_n^{k_n(\sigma)} \cdot \text{Tr}(\sigma|_{H^i(F(X,n)/G)}) =$$

(by Lefschetz's theorem) = 
$$\sum_{\sigma \in S_n} (-1)^{|\sigma|} p_1^{k_1(\sigma)} \cdot \ldots \cdot p_n^{k_n(\sigma)} \cdot \chi([F(X,n)/G]^{\sigma}) =$$

(by Lemma 2) = 
$$\frac{1}{|G|} \sum_{g \in G} \sum_{\sigma \in S_n} (-1)^{|\sigma|} p_1^{k_1(\sigma)} \cdot \dots \cdot p_n^{k_n(\sigma)} \cdot \chi([F(X, n)]^{g^{-1}\sigma}).$$

If for a tuple  $\mathbf{y} \in F(X, n)$  we have  $\sigma \mathbf{y} = g \mathbf{y}$ , then the action of (g) at this tuple has  $k_j(\sigma)$  cycles of length j. Every cycle of length j corresponds to a point in  $X_j(g)/(g)$ , so we can assign a point in  $\prod_j F(X_j(g)/(g), k_j)$ .

It rests to count the number of possible orderings on  $\mathbf{y}$  with given  $\sigma$  and set of orbits. One should establish a correspondence between cycles in  $\sigma$  and orbits, what can be done in  $\prod k_i!$  ways.  $\square$ 

Following [12], we give a "fibered version" of Theorem 1.

Let  $f: E \to B$  be a family of varieties  $V_b = f^{-1}(b)$ , which are equipped with the action of discrete automorphism groups  $G_b$ , depending of a point b on the base B. Let us define a collection of spaces  $B_{\underline{k}}$  in the following way:

$$B_{k_1,\dots,k_n} := \{(b,g)|b \in B, g \in G_b, \chi((V_b)_i(g)) = ik_i\}.$$
(3)

These spaces carry natural orbifold structure – a group  $G_b$  corresponds to a point (b, g).

Consider also a collection of configuration spaces  $E_n/G$  as fibrations over B with fibers  $F(V_b, n)/G_b$ .

**Theorem 2** The following equation holds:

$$\sum_{n=0}^{\infty} t^n \chi^{S_n}(E_n/G) = \sum_{\underline{k}} \chi^{orb}(B_{\underline{k}}) \cdot \prod_{j=1}^{\infty} (1 + t^j p_j)^{k_j}.$$

**Proof**. From the Theorem 1 and the additivity of the Euler characteristic we have

$$\sum_{n=0}^{\infty} t^n \chi^{S_n}(E_n/G) = \int_B \frac{1}{|G_b|} \sum_{g \in G_b} \prod_{k=1}^{\infty} (1 + p_k t^k)^{\frac{\chi((V_b)_k(g))}{k}} d\chi. \tag{4}$$

Now let us collect the summands together. Let  $N_{\underline{k}}(b)$  denote the number of elements  $g \in G_b$  such that for all i one has  $\chi((V_b)(g)) = ik_i$ . We can rewrite (4) as follows:

$$\sum_{n=0}^{\infty} t^n \chi^{S_n}(E_n/G) = \sum_{k} \prod_{i=1}^{\infty} (1 + p_i t^i)^{k_i} \cdot \int_B \frac{N_{\underline{k}}(b)}{|G_b|} d\chi.$$

On the other hand, consider a projection of the space  $B_{k_1,\ldots,k_n}$  to the base B. Its fiber over a point B consists of exactly  $N_{k_1,\ldots,k_n}(b)$  elements of  $G_b$ , so by the Fubini's formula we have

$$\chi^{orb}(B_{k_1,\dots,k_n}) = \int_B \frac{N_{\underline{k}}(b)}{|G_b|} d\chi.$$

**Corollary 1** ([12]) Let  $\mathcal{M}_g(k_1,\ldots,k_n)$  be the moduli space of pairs  $(C,\tau)$  where C is a genus g curve and  $\tau$  is an automorphism of C such that  $\chi(C_i(\tau)) = ik_i$  for all i. Then the following equation holds:

$$\sum_{n=0}^{\infty} t^n \chi^{S_n}(\mathcal{M}_{g,n}) = \sum_k \chi^{orb}(\mathcal{M}_g(k_1, \dots, k_n)) \cdot \prod_{j=1}^{\infty} (1 + t^j p_j)^{k_j}.$$
 (5)

## 3 Moduli spaces of curves

We start with the combinatoral computation from [13].

**Lemma 3** Let  $\zeta$  be a primitive  $k^{th}$  root of unity and l|k. Then

$$\sum_{(r,k)=l} \zeta^r = \mu(\frac{k}{l}),$$

where  $\mu$  denotes the Moebius function.

**Proof.** If (r, k) = l, then  $\zeta^r$  is a primitive root of unity of degree  $\frac{k}{l}$ . Therefore the value of the left hand side depends only of  $\frac{k}{l}$ :

$$\sum_{(r,k)=l} \zeta^r = g(\frac{k}{l}).$$

We have for all k

$$\sum_{l|k} g(\frac{k}{l}) = \sum_{l|k} \sum_{(r,k)=l} \zeta^r = \sum_{r=0}^{k-1} \zeta^r = \delta_{k,1},$$

so by the Moebius inversion formula we have  $g(k/l) = \mu(k/l)$ .  $\square$ 

**Lemma 4** Let l|k, d|k and let  $\zeta$  be a primitive  $d^{th}$  root of unity. Then

$$\sum_{0 \le r < k, (r,k) = l} \zeta^r = \mu(\frac{d}{(d,l)}) \frac{\varphi(k/l)}{\varphi(d/(d,l))}.$$
 (6)

**Proof**. First, remark that as values of the expression  $\zeta^r$  we'll receive every of  $\varphi(\frac{d}{(d,l)})$  primitive roots of unity of degree  $\frac{d}{(d,l)}$ . The sum of these roots is, according to lemma 3, equal to  $\mu(\frac{d}{(d,l)})$ .

Second, we have  $\varphi(\frac{k}{l})$  summands in the left hand side of (6), so every of these roots will appear  $\frac{\varphi(k/l)}{\varphi(d/(d,l))}$  times.  $\square$ 

Let

$$c(k, l, d) = \mu(\frac{d}{(d, l)}) \frac{\varphi(k/l)}{\varphi(d/(d, l))}.$$

Lemma 5 ([13]) Let

$$N(k; l_1, \dots, l_s) = |\{(r_1, \dots, r_s) \in (\mathbb{Z}/k\mathbb{Z})^s | r_1 + \dots + r_s \equiv 0 \pmod{k}, (r_i, k) = l_i\}|.$$
(7)

Then

$$N(k; l_1, \dots, l_s) = \frac{1}{k} \sum_{d|k} \varphi(d) \prod_{i=1}^s c(k, l_i, d).$$
 (8)

**Proof**. Remark that

$$\frac{1}{k} \sum_{\zeta^k = 1} \zeta^r = \begin{cases} 1 & , r \equiv 0 \pmod{k} \\ 0 & , \text{ otherwise} \end{cases}$$

Therefore

$$N(k; l_1, \dots, l_s) = \frac{1}{k} \sum_{\zeta^k = 1} \sum_{(r_i, k) = l_i} \zeta^{r_1 + \dots + r_s} =$$

$$\frac{1}{k} \sum_{\zeta^{k}=1} \prod_{i=1}^{s} \sum_{(r_{i},k)=l_{i}} \zeta^{r_{i}} = \frac{1}{k} \sum_{d|k} \varphi(d) \prod_{i=1}^{s} c(k, l_{i}, d).$$

The last equation is true since among all roots  $\zeta$  there are  $\varphi(d)$  primitive roots of degree d, and we can use the equation (6) for them.  $\square$ 

**Example 1** Let p be a prime number, then

$$N(p;\underbrace{1,\ldots,1}_{s}) = \frac{1}{p}(1 \cdot c(p,1,1)^{s} + (p-1) \cdot c(p,1,p)^{s}) = \frac{1}{p}((p-1)^{s} + (-1)^{s}(p-1)).$$
(9)

This formula can be checked explicitly using the recurrence relation

$$N(p; \underbrace{1, \dots, 1}_{s}) = (p-1)^{s-1} - N(p; \underbrace{1, \dots, 1}_{s-1}).$$

The value of  $r_s$  is uniquely determined modulo p by  $r_1, \ldots, r_{s-1}$ , and it vanishes, if and only if  $(r_1, \ldots, r_{s-1})$  appears in the count of  $N(p; \underbrace{1, \ldots, 1})$ .

Let us return to the moduli spaces. From the Corollary 1 we conclude that we have to compute the orbifold Euler characteristics of the configuration spaces  $\mathcal{M}_g(k_1,\ldots,k_n)$  of pairs  $(C,\tau)$ , where C is a smooth connected curve,  $\tau$  is its automorphism and for all j  $\chi(C_j(\tau)) = jk_j$ .

Let  $\tau$  be an automorphism of a genus g curve C such that  $\chi(C_j(\tau)) = jk_j$ . Note that

$$\sum_{j=1}^{n} jk_j = \chi(C) = 2 - 2g.$$

Cosider the quotient  $C_1 = C/\tau$ . It is a smooth curve of some genus h, and the Riemann-Hurwitz formula immediately gives its Euler characteristic as

$$\chi(C_1) = 2 - 2h = \sum_{j=1}^{n} k_j.$$

On the curve  $C_1$  we have  $s = \sum_{j=1}^{n-1} k_j$  marked points, outside of them the projection of C to  $C_1$  is the non-ramified covering of order n. The number n is also the order of  $\tau$ , so j|n, if  $k_j \neq 0$ .

Define the numbers  $l_1, \ldots, l_s$  as

$$l_1 = \dots = l_{k_1} = 1, \dots, l_{s-k_{n-1}+1} = \dots = l_s = n-1.$$
 (10)

**Lemma 6** ([13]) Let us fix the quotient curve  $C_1$ , positions of s marked points on it and integers  $k_1, \ldots, k_n$ . Then the equivariant Euler characteristic

of the set of pairs  $(C, \tau)$  (where C is connected) providing such set of data equals to

$$\frac{1}{n}n^{2h}\prod_{p|(l_1,\dots,l_s)}(1-p^{2h})\cdot N(n;l_1,\dots,l_s)$$

**Proof.** Let us fix a generic point x on the curve  $C_1$  and also fix one of its preimages  $y_1$ . The action of  $\tau$  will give a natural enumeration on the set of preimages  $y_1, \ldots, y_n$ . Since all of them are equivalent, this choice will multiply the number by n.

To every of marked points p we assign the monodromy around a loop beginning at x, which can be identified with an element r(p) of  $\mathbb{Z}/n\mathbb{Z}$ . If p corresponds to the orbit of length j, we have (r(p), n) = j, so we can enumerate points  $p_i$  in such a way that  $(r(p_i), n) = l_i$ . Also we have  $\sum_p r_p = 0 \pmod{n}$ , since the composition of monodromies around ramification points is trivial.

Consider the set of monodromies  $a_1, \ldots, a_{2h}$  around the basis cycles in  $H_1(C_1)$ . The set of residues  $a_1, \ldots, a_{2h}, r_1, \ldots, r_s$  defines the topology of the ramified covering completely, but we should take into account that the covering C should be a connected curve. The connectedness of the covering is equivalent to the condition

$$g.c.d(a_1, \ldots, a_{2h}; r_1, \ldots, r_s; n) = 1.$$

Therefore for fixed  $r_1, \ldots, r_s$  we must count the number of (2h)-tuples in  $(\mathbb{Z}/n\mathbb{Z})^{2h}$  whose greatest common divisor is coprime to  $(l_1, \ldots, l_s)$ . This number is equal to

$$n^{2h} \prod_{p|(l_1,\dots,l_s)} (1 - p^{2h}).$$

**Theorem 3** The generating function for the  $S_n$ -equivariant Euler characteristics of  $\mathcal{M}_{g,n}$  has a form

$$\sum_{n=0}^{\infty} t^n \chi^{S_n}(\mathcal{M}_{g,n}) = \sum_{\underline{k}} c_{k_1,\dots,k_n} \prod_{j \le n} (1 + p_j t^j)^{k_j}, \tag{11}$$

where  $p_i$  are Newton symmetric polynomials,

$$\sum_{j \le n} j \cdot k_j = 2 - 2g$$

and the coefficients  $c_{k_1,\ldots,k_n}$  are defined in the following way. Let

$$h = \frac{1}{2}(1 - \sum_{j \le n} k_j), s = \sum_{j \le n} k_j,$$

and  $l_1, \ldots l_s$  are defined by (10). Then

$$c_{k_1,\dots,k_n} = \chi^{orb}(\mathcal{M}_{h,s}) n^{2h} \prod_{p|(l_1,\dots,l_s)} (1-p^{2h}) \cdot \frac{1}{k_1! k_2! \dots k_{n-1}! \cdot n} N(n; l_1,\dots,l_s).$$
(12)

where

$$\chi^{orb}(\mathcal{M}_{h,s}) = (-1)^s \frac{(2g-1) \cdot B_{2g}}{(2g-3)!}$$
(13)

is the orbifold Euler characteristic of  $\mathcal{M}_{h,s}$  ( $B_{2g}$  are Bernoulli numbers), and  $N(n; l_1, \ldots, l_s)$  is the number of solutions of the equation

$$r_1 + \ldots + r_s \equiv 0 \pmod{n} \tag{14}$$

such that  $g.c.d.(n, r_i) = l_i$ .

**Proof**. By the equation (5) we have

$$\sum_{n=0}^{\infty} t^n \chi^{S_n}(\mathcal{M}_{g,n}) = \sum_k \chi^{orb}(\mathcal{M}_g(k_1, \dots, k_n)) \cdot \prod_{j=1}^{\infty} (1 + t^j p_j)^{k_j}.$$

Now consider the moduli space  $\mathcal{M}_g(k_1,\ldots,k_n)$  of pairs  $(C,\tau)$ . To such a pair we can associate a ramified covering over a quotient curve  $C_1 = C/\tau$ , which is defined by:

- A curve  $C_1$  and s marked points on it. The corresponding orbifold Euler characteristic is equal to  $\chi^{orb}(\mathcal{M}_{h,s})$ .
- Subdivision of marked points into groups of size  $k_1, \ldots, k_{n-1}$ . This can be done in  $k_1!k_2!\ldots k_{n-1}!$  ways so we have the factor  $\frac{1}{k_1!k_2!\ldots k_{n-1}!}$ .
- Number of coverings with fixed marked points: by Lemma 6 the corresponding orbifold Euler characteristic equals to

$$\frac{1}{n}n^{2h}\prod_{p|(l_1,\ldots,l_s)}(1-p^{2h})\cdot N(n;l_1,\ldots,l_s).$$

This completes the proof.  $\Box$ 

## Appendix: genus 2 curves

The generating function for the  $S_n$ -equivariant Euler characteristics of the moduli spaces of genus 2 curves with marked points has a form ([11]):

$$\sum_{n=0}^{\infty} t^n \chi^{S_n} (\mathcal{M}_{2,n}) = \frac{-1}{240} (1 + p_1 t)^{-2} - \frac{1}{240} (1 + p_1 t)^6 (1 + p_2 t^2)^{-4} +$$

$$+ \frac{2}{5} (1 + p_1 t)^3 (1 + p_5 t^5)^{-1} + \frac{2}{5} (1 + p_1 t) (1 + p_2 t^2) (1 + p_5 t^5) (1 + p_{10} t^{10})^{-1} +$$

$$+ \frac{1}{6} (1 + p_1 t)^2 (1 + p_2 t^2) (1 + p_6 t^6)^{-1} - \frac{1}{12} (1 + p_1 t)^4 (1 + p_3 t^3)^{-2} -$$

$$- \frac{1}{12} (1 + p_2 t^2)^2 (1 + p_3 t^3)^2 (1 + p_6 t^6)^{-2} + \frac{1}{12} (1 + p_1 t)^2 (1 + p_2 t^2)^{-2} +$$

$$+ \frac{1}{4} (1 + p_1 t)^2 (1 + p_4 t^4) (1 + p_8 t^8)^{-1} - \frac{1}{8} (1 + p_1 t)^2 (1 + p_2 t^2)^2 (1 + p_4 t^4)^{-2}.$$

Let us compare the coefficients with the theorem 3.

 $1)(1+p_1t)^{-2}$ . We have h=2, s=0, n=1, so the coefficient equals to

$$\chi^{orb}(\mathcal{M}_{2,0}) = \frac{-1}{240}.$$

Remark that for general g we'll have analogous "identity" summand

$$\chi^{orb}(\mathcal{M}_{g,0}) \cdot (1+p_1t)^{2-2g}$$
.

 $2)(1+p_1t)^6(1+p_2t^2)^{-4}$ . We have  $h=0, s=k_1=6, n=2$ . Therefore the coefficient equals to

$$\frac{\chi^{orb}(\mathcal{M}_{0,6})}{6! \cdot 2} = \frac{-1}{240}.$$

Remark that for general g we'll have a "hyperelliptic" summand

$$\frac{\chi^{orb}(\mathcal{M}_{0,2g+2})}{(2g+2)! \cdot 2} (1+p_1t)^{2g+2} (1+p_2t^2)^{-2g} = \frac{-(2g-1)!}{(2g+2)! \cdot 2} (1+p_1t)^{2g+2} (1+p_2t^2)^{-2g} = \frac{-1}{4g(2g+1)(2g+2)} (1+p_1t)^{2g+2} (1+p_2t^2)^{-2g}.$$

 $3)(1+p_1t)^3(1+p_5t^5)^{-1}$ . We have  $h=0, s=k_1=3, n=5$ . The coefficient equals to

$$\frac{\chi^{orb}(\mathcal{M}_{0,3}) \cdot N(5;1,1,1)}{3! \cdot 5} = \frac{1 \cdot 12}{6 \cdot 5} = \frac{2}{5}.$$

We used the equation

$$N(5;1;1;1) = \frac{1}{5} \sum_{d|5} \varphi(d) (\mu(d) \frac{\varphi(5)}{\varphi(d)})^3 = \frac{1}{5} (64 - 4) = 12.$$

 $4(1+p_1t)(1+p_2t^2)(1+p_5t^5)(1+p_{10}t^{10})^{-1}$ . We have h=0, s=3, n=10. The coefficient equals to

$$\frac{\chi^{orb}(\mathcal{M}_{0,3}) \cdot N(10; 1, 2, 5)}{10} = \frac{1 \cdot 4}{10} = \frac{2}{5}.$$

We used the equation N(10; 1, 2, 5) = 4, since there are 4 triples (1, 4, 5), (3, 2, 5), (7,8,5),(9,6,5) with the desired property

$$(r_1, 10) = 1, (r_2, 10) = 2, (r_3, 10) = 5, r_1 + r_2 + r_3 \equiv 0 \pmod{10}.$$

 $(5)(1+p_1t)^2(1+p_2t^2)(1+p_6t^6)^{-1}$ . We have h=0, s=3, n=6, the coefficient equals to

$$\frac{\chi^{orb}(\mathcal{M}_{0,3}) \cdot N(6; 1, 1, 2)}{2! \cdot 6} = \frac{1}{6},$$

since N(6; 1, 1, 2) = 2 is represented by triples (1, 1, 4), (5, 5, 2).

 $6)(1+p_1t)^4(1+p_3t^3)^{-2}$ . We have h=0, s=4, n=3, the coefficient equals to

$$\frac{\chi^{orb}(\mathcal{M}_{0,4}) \cdot N(3;1,1,1,1)}{4! \cdot 3} = \frac{-1 \cdot 6}{24 \cdot 3} = \frac{-1}{12}.$$

Here we can use the formula (9):  $N(3; 1, 1, 1, 1) = \frac{1}{3}(2^4 + 2) = 6$ .  $7)(1 + p_2t^2)^2(1 + p_3t^3)^2(1 + p_6t^6)^{-2}$ . We have h = 0, s = 4, n = 6, the coefficient equals to

$$\frac{\chi^{orb}(\mathcal{M}_{0,4}) \cdot N(6; 2, 2, 3, 3)}{2! \cdot 2! \cdot 6} = \frac{-1 \cdot 2}{24} = \frac{-1}{12},$$

since N(6; 1, 1, 2) = 2 is represented by quadruples (2, 4, 3, 3), (4, 2, 3, 3).

 $8)(1+p_1t)^2(1+p_2t^2)^{-2}$ . We have h=1,s=2,n=2, the coefficient equals to

$$\frac{\chi^{orb}(\mathcal{M}_{1,2}) \cdot N(2;1,1) \cdot 2^2}{2! \cdot 2} = \frac{1}{12},$$

since N(2; 1, 1) = 1. This case is of special interest – it is the only one where the quotient curve has positive genus h, so additional multiplier  $n^{2h} = 2^2$  appears.

9) $(1 + p_1t)^2(1 + p_4t^4)(1 + p_8t^8)^{-1}$ . We have h = 0, s = 3, n = 8, the coefficient equals to

$$\frac{\chi^{orb}(\mathcal{M}_{0,3}) \cdot N(8; 1, 1, 4)}{2! \cdot 8} = \frac{1}{4},$$

since N(8; 1, 1, 4) = 4 is represented by triples (1, 3, 4), (3, 1, 4), (5, 7, 4), (7, 5, 4).  $10)(1 + p_1t)^2(1 + p_2t^2)^2(1 + p_4t^4)^{-2}$ . We have h = 0, s = 4, n = 4, the coefficient equals to

$$\frac{\chi^{orb}(\mathcal{M}_{0,4}) \cdot N(4;1,1,2,2)}{2! \cdot 2! \cdot 4} = \frac{-1}{8},$$

since N(4; 1, 1, 2, 2) = 2 is represented by quadruples (1, 3, 2, 2), (3, 1, 2, 2).

### References

- [1] J. Bergström. Cohomologies of moduli spaces of curves of genus three via point counts. arXiv:math/0611815.
- [2] J. Bergström, G. van der Geer. The Euler characteristic of local systems on the moduli of curves and abelian varieties of genus three. arXiv:0705.0293
- [3] J. Bergström, O. Tommasi. The rational cohomology of  $\overline{\mathcal{M}}_4$ . Math. Ann. 338 (2007), no. 1, 207–239.
- [4] G. Bini, J. Harer. Euler Characteristics of Moduli Spaces of Curves. arXiv:math/0506083.
- [5] G. Bini, G. Gaiffi, M. Polito. A formula for the Euler characteristic of  $\overline{\mathcal{M}}_{2,n}$ .

- [6] E. Getzler. Mixed Hodge structures of configuration spaces. arXiv:math.AG/9510018
- [7] E. Getzler. Resolving mixed Hodge modules on configuration spaces. Duke Math. J. 96 (1999), no. 1, 175–203.
- [8] E. Getzler. Euler characteristics of local systems on  $\mathcal{M}_2$ . Compositio Math. 132 (2002), 121–135.
- [9] E. Getzler, M. Kapranov. Modular operads. Compositio Math. 110 (1998), 62–126.
- [10] E. Getzler, E. Looijenga. The Hodge polynomial of  $\overline{\mathcal{M}}_{3,1}$ .
- [11] E. Gorsky. On the  $S_n$  equivariant Euler characteristic of  $\mathcal{M}_{2,n}$ . arXiv:math.AG/0707.2662.
- [12] E. Gorsky. On the  $S_n$ -equivariant Euler characteristic of moduli spaces of hyperelliptic curves.
- [13] J. Harer, D. Zagier. The Euler characteristic of the moduli space of curves. Invent. math. 85, 457-485 (1986).
- [14] C. Faber, G. van der Geer. Sur la cohomologie des systemes locaux sur les espaces des modules des courbes de genre 2 et des surfaces abeliennes. arXiv:math/0305094.
- [15] O. Tommasi. Rational cohomology of the moduli space of genus 4 curves. arXiv:math/0312055.
- [16] O. Tommasi. Rational cohomology of  $\mathcal{M}_{3,2}$ . arXiv:math/0611053

Laboratoire J.-V. Poncelet (UMI 2615).

E. mail: gorsky@mccme.ru